ON THE INVARIANTS OF DIFFERENTIAL FORMS OF DEGREE HIGHER THAN TWO[*]

BY

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§ 1. *Introduction.*

The following determination of the number of invariants of homogeneous differential forms of higher degree is, in notation and method, an extension of the work in my paper on the invariants of quadratic differential forms.[†]

The result may be stated in the following

THEOREM.—*The number of invariants of order μ for the general homogeneous differential form of degree m in n variables is*

$$\frac{(n+\mu-1)!}{(n-1)!\,\mu!}\left\{\frac{(n+m-1)!}{(n-1)!\,m!}-\frac{n(n+\mu)}{\mu+1}\right\}, \qquad m > 2.$$

It may be noted that, save for the exceptional cases

$$n \gneqq 2, \ \mu = 0, 1, \text{ or } 2; \qquad \text{and} \qquad n = 2, \ \mu = 3;$$

this expression gives also the number of invariants of quadratic differential form $(m = 2)$.[‡]

§ 2. *Differential equations of the problem.*

The form of degree m in n variables, $x_1 \cdots x_n$, may be written

(1)
$$\phi \equiv \sum_{i_1=1}^{n} \sum_{i_2=1}^{n} \cdots \sum_{i_m=1}^{n} a_{i_1 i_2 \ldots i_m}\, dx_{i_1}\, dx_{i_2} \cdots dx_{i_m}$$

where

$$a_{i_1 i_2 \ldots i_m} = a_{k_1 k_2 \ldots k_m},$$

if $i_1, \cdots i_m$; $k_1, \cdots k_m$ are any two permutations of the same set of m indices.

It will be assumed throughout that the discriminant of the form ϕ, i. e., the resultant of the n equations

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(2)
$$\frac{\partial \phi}{\partial (dx_r)} = 0, \qquad (r = 1, 2, \cdots n),$$

does not vanish.

The form is to be subjected to the infinite group of all point transformations generated by the infinitesimal transformation

(3)
$$Xf \equiv \sum_{r=1}^{n} \xi_r(x_1 \cdots x_n) \frac{\partial f}{\partial x_r}.$$

To obtain the equations satisfied by the invariants, we must "extend" the group Xf to the coefficients a and their derivatives. As in the similar work with quadratic forms, we use the notation

$$\alpha_{i_1 i_2 \ldots i_m} = X(a_{i_1 i_2 \ldots i_m}),$$

(4)
$$p_{i_1 i_2 \ldots i_m} = \frac{\partial f}{\partial (a_{i_1 i_2 \ldots i_m})},$$

Then we have
$$\xi_{r\,l} = \xi_r = \frac{\partial \xi_r}{\partial x_l}.$$

$$X\phi = \sum_{i_1=1}^{n} \sum_{i_2=1}^{n} \cdots \sum_{i_m=1}^{n} \alpha_{i_1 i_2 \ldots i_m}\, dx_{i_1}\, dx_{i_2} \cdots dx_{i_m}$$

(5)
$$+ \sum_{i_1=1}^{n} \sum_{i_2=1}^{n} \cdots \sum_{i_m=1}^{n} a_{i_1 i_2 \ldots i_m} \sum_{l=1}^{m} dx_{i_1} \cdots dx_{i_{l-1}} dx_{i_{l+1}} \cdots dx_{i_m} \sum_{r=1}^{n} \xi_{i_l} dx_r$$

$$= \sum_{i_1=1}^{n} \sum_{i_2=1}^{n} \cdots \sum_{i_m=1}^{n} dx_{i_1}\, dx_{i_2} \cdots dx_{i_m} \left\{ \alpha_{i_1 i_2 \ldots i_m} + \sum_{r=1}^{n} \sum_{l=1}^{m} a_{i_1 \ldots i_{l-1} r\, i_{l+1} \ldots i_m} \xi_r \atop i_l \right\}.$$

Hence

(6)
$$- \alpha_{i_1 i_2 \ldots i_m} = \sum_{r=1}^{n} \sum_{l=1}^{m} a_{i_1 i_2 \ldots i_{l-1} r\, i_{l+1} \ldots i_m} \xi_r \atop i_l.$$

The equations for the invariants of order zero are obtained by equating to zero the coefficients of the $\xi_{r|i_l}$ in

(7)
$$X_0 f \equiv \sum_{i_1=1}^{n} \sum_{i_2=1}^{i_1} \cdots \sum_{i_m=1}^{i_{m-1}} \alpha_{i_1 i_2 \ldots i_m} p_{i_1 i_2 \ldots i_m}.$$

§ 3. *Enumeration of the equations.*

There are n^2 of the quantities $\xi_{r|s}$, hence n^2 equations. The number of variables $p_{i_1 i_2 \ldots i_m}$ is that of the number of distinct terms in the homogeneous polynomial of degree m in n variables, namely *

(8)
$$(n, m) \equiv \frac{n(n+1)\cdots(n+m-1)}{m!} = \frac{(n+m-1)!}{(n-1)!\, m!}.$$

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* Cf. Chrystal, *Algebra*, vol. 2, p. 12.

We suppose in what follows that

$$m > 2, \qquad n \gtreqqless 2.$$

Now

$$(n, m + 1) = \frac{n + m}{m + 1}(n, m),$$

and

$$\frac{n + m}{1 + m} > 1.$$

$$\therefore (n, m + 1) > (n, m),$$

$$\therefore (n, m) > (n, 3), \qquad m > 3,$$

$$\therefore (n, m) - n^2 > (n, 3) - n^2, \qquad m > 3.$$

But

$$(n, 3) - n^2 = 0, \qquad n = 2,$$

and

$$(n, 3) - n^2 > 0, \qquad n > 2.$$

For

$$(n, 3) - n^2 = \frac{n(n-1)(n-2)}{6},$$

(9)

$$\therefore (n, m) - n^2 = 0, \qquad n = 2, m = 3,$$

and

$$(n, m) - n^2 > 0, \qquad n > 2, m \gtreqqless 3.$$

Hence, whether the equations of order zero are independent or not, there are certainly, except in the case of the binary cubic, for which $n = 2$, $m = 3$, invariants of order zero; and, if the equations of order zero are all independent, there are precisely

$$(n, m) - n^2$$

invariants of that order.

Consider now the equations of higher order. In passing from the $(\mu - 1)$th extension to the μth we add a number of equations equal to the number of derivatives of order $\mu + 1$ of the n functions ξ_r, and a number of variables equal to the number of derivatives of order μ of the (n, m) functions $a_{i_1 i_2 \ldots i_m}$. These numbers are, respectively,

$$n(n, \mu + 1) = \frac{n(n + \mu)}{\mu + 1}(n, \mu),$$

and

$$(n, m) \cdot (n, \mu).$$

Hence, if the equations of orders $\mu - 1$ and μ are all independent, the number of invariants of order $\equiv \mu$ is

$$(10) \qquad \left\{ (n, m) - \frac{n(n + \mu)}{\mu + 1} \right\} (n, \mu).$$

§ 4. *Independence of the equations.*

In the enumeration of the invariants of the quadratic differential forms the independence of the equations was proved step by step in the following manner.

First, it was shown that the equations of order μ differ from those of order $\mu - 1$ in that, first, certain terms are added to those equations, and second, certain equations are annexed to the system. These annexed equations were called the *final* equations of order μ. It was shown that the variables in these equations appear in the equations of no lower order.

Second, it was shown that if all the equations of order $\mu - 1$ are independent, and, if the final equations of order μ are independent, then all the equations of order μ are independent.

Third, it was shown that if the *final* equations of order $\mu - 1$ are all independent, then the *final* equations of order μ must be so likewise.

Fourth, it was shown that there exists an order μ for which the final equations are independent, and an order μ_1 for which all the equations are independent.

In the present problem the proof follows the same lines. Furthermore, the demonstration of the first three steps in the proof follows almost word for word the demonstration already given for quadratic forms. It will, therefore, be omitted here. In the case of quadratic forms, the two parts of the fourth step required separate demonstration. In the present case the matter is simplified in that we are able to prove the following

THEOREM.— *The equations of order zero are, for forms of degree $m \geqq 3$ in $n \geqq 2$ variables, all independent.*

As the equations of order zero are likewise the final equations of that order, the entire fourth step of the proof is covered by this theorem. In the proof it is convenient, in order to avoid disturbing numerical factors, to write the form as follows:

$$(11) \qquad \phi \equiv \sum_{i_1=1}^{n} \sum_{i_2=1}^{i_1} \cdots \sum_{i_m=1}^{i_{m-1}} b_{i_1 i_2 \ldots i_m} \, dx_{i_1} dx_{i_2} \cdots dx_{i_m}.$$

Then

$$(12) \quad X\phi = \sum_{i_1=1}^{n} \sum_{i_2=1}^{i_1} \cdots \sum_{i_m=1}^{i_{m-1}} (X b_{i_1 i_2 \ldots i_m}) \, dx_{i_1} dx_{i_2} dx_{i_m} + \sum_{s=1}^{n} \frac{\partial \phi}{\partial (dx_s)} \sum_{r=1}^{n} \xi_s \, dx_r.$$

The coefficient of $\xi_{s|r}$ is $\{ \partial \phi / \partial (dx_s) \} \, dx_r$.

Now the equations for the invariants of order zero are obtained by collecting those terms of the second part of the above expression for $X\phi$ which are multiplied by each of the products $dx_{i_1} \cdots dx_{i_m}$, replacing $dx_{i_1} \cdots dx_{i_m}$ by

$$(13) \qquad\qquad q_{i_1 \ldots i_m} \equiv \frac{\partial f}{\partial (b_{i_1 \ldots i_m})},$$

rearranging the result so as to collect the terms forming the coefficients of the n^2 quantities $\xi_{s|r}$, and then equating these coefficients to zero. Hence these equations may be obtained if in the equations

$$(14) \qquad\qquad \frac{\partial \phi}{\partial (dx_s)} dx_r = 0 , \qquad\qquad (r, s = 1, 2, \cdots n),$$

we replace the product $dx_{i_1} \cdots dx_{i_m}$ by the symbol $q_{i_1 \ldots i_m}$. The matrix of the coefficients, therefore, which is all that is of importance in the independence proof, is the same as the matrix of the coefficients in equations (14).

Consider, next, the discriminant of the form ϕ. It is the resultant of the n equations

$$(15) \qquad\qquad \frac{\partial \phi}{\partial (dx_r)} = 0, \qquad\qquad (r = 1, 2 \cdots n),$$

and may be written as a determinant which includes the matrix of n^2 rows and (n, m) columns arising from the equations

$$(16) \qquad\qquad \frac{\partial \phi}{\partial (dx_s)} dx_r = 0, \qquad\qquad (r, s = 1, 2, \cdots n),$$

together with additional rows and columns.

But in the columns annexed to these n^2 rows the elements are all zero.

Suppose the equations of order zero were not all independent. Then *all* the determinants of order n^2 formed from the matrix of (14) vanish. But in the resultant all other minors of order n^2 formed from these n^2 rows contain at least one column of zeros and therefore vanish. Hence *all* minors of order n^2 formed from these n^2 rows of the resultant vanish. Therefore the resultant itself vanishes. But this resultant is the discriminant of the form, and by hypothesis it does not vanish. Hence the supposition that the equations of order zero are not all independent leads to a contradiction and the theorem is proved.

It follows, then, that the equations of all orders are independent and that, therefore, the number of invariants of order μ is precisely the excess of variables added over equations added in passing from order $\mu - 1$ to order μ, namely

$$(17) \qquad\qquad \left\{ (n, m) - \frac{n(n + \mu)}{\mu + 1} \right\} (n, \mu).$$

§ 5. *Simultaneous forms.*

The extension to simultaneous forms is obvious.   There are for p forms

$$(18) \qquad (n, \mu) \left\{ p(n, m) - \frac{n(n+\mu)}{\mu+1} \right\} = (n, \mu) \left\{ p \left[(n, m) - \frac{n(n+\mu)}{\mu+1} \right] \right.$$

$$\left. + (p-1) \frac{n(n+\mu)}{\mu+1} \right\}$$

invariants of order μ.   Of these

$$(19) \qquad p \left[(n, m) - \frac{n(n+\mu)}{\mu+1} \right] (n, \mu)$$

are invariants of the p separate forms, while the remaining

$$(20) \qquad (p-1) \frac{n(n+\mu)}{\mu+1} (n, \mu)$$

are true simultaneous invariants which may be taken as invariants of the $p - 1$ pairs of forms obtained by associating any given form with the remaining $p - 1$ in succession.

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